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and $N'$ is also the foot of a perpendicular on $BC$ from $P$, the foot of a perpendicular from $A$ to $BDC$.

Since for different tetrahedra $P$ becomes any point in plane $BCD$, the following theorem is obtained (Fig. 3): If from a point ($P$) perpendiculars be drawn to the sides of a triangle ($BCD$) produced if necessary, and their feet ($L'M'N'$) be joined to the opposite vertices, the three intersections of these lines with the parallels to the sides through the median point ($LMN$) and the median point ($A'$) lie on a circle whose center divides the join of the circum-center of the original triangle and the center of mean position of the three vertices of the triangle and the arbitrary point externally in the ratio 1 : 4. The second part of the theorem follows from the fact that $B'C'D'$ and $BCD$ (Fig. 2) are similarly situated with respect to the center of mass of the four vertices of the tetrahedron, and the orthogonal projection of the circum-center of $A'B'C'D'$ on plane $BCD$ coincides with that of the circum-center of triangle $B'C'D'$.

# QUESTIONS AND DISCUSSIONS.

Edited by W. A. Hurwitz, Cornell University, Ithaca, N. Y.

## NEW QUESTIONS.

**37.** Criticise the following as fundamental definitions of elementary geometry:

A plane surface is the limit approached by a finite portion of the surface of a sphere as the radius increases without limit.

A straight line is the limit approached by a finite portion of the circumference of a circle as the radius increases without limit.

**38.** As the several courses in secondary and collegiate mathematics are now taught there is a noticeable difference of treatment with respect to the relative emphasis placed on logical accuracy and on development of technique. In elementary algebra, technique predominates, while in plane and solid geometry and advanced algebra logic is more strongly stressed. Trigonometry, analytic geometry, and the calculus are less easily classified; but there is at least a general tendency to emphasize logic in analytic geometry and technique in the calculus. It is suggested that a general appraisal of the reasons for this difference in treatment, its value, and possible alterations, would be helpful.

## DISCUSSIONS.

### THE EXISTENCE OF CUSPS ON THE EVOLUTE AT POINTS OF MAXIMUM AND MINIMUM CURVATURE ON THE BASE CURVE.

By G. H. Light, University of Colorado.

The purpose of this paper is to show that whenever a curve $F(x, y) = 0$ has a point of maximum or minimum curvature the evolute has at the corresponding point a cusp at which the curvature is infinite.[1] The assumption is made that at the point on the curve $T$ under consideration $F_y \neq 0$.

Hence the value of $dy/dx$ may be written

$$dy/dx = - F_x/F_y,$$

from which one easily obtains

$$d^2y/dx^2 = - B/F_y^3,$$

where

$$B = F_x^2F_{yy} - 2F_xF_yF_{xy} + F_y^2F_{xx}.$$

By definition, the curvature has the following value;

$$k = - \frac{B}{(F_x^2 + F_y^2)^{3/2}}.$$

Hence at a critical value its derivative must vanish; *i. e.*, $dk/dx = 0$. Performing the necessary work, it is found that

$$\frac{dk}{dx} = - \frac{(C - 3D)}{F_y(F_x^2 + F_y^2)^{5/2}},$$

where $C$ and $D$ have the values:

$$C = (F_x^2 + F_y^2)(F_y^3F_{xxx} - 3F_y^2F_xF_{xxy} + 3F_yF_x^2F_{xyy} - F_x^3F_{yyy}),$$

$$\begin{aligned} D = {} & (F_xF_yF_{xx} + F_y^2F_{xy} - F_x^2F_{xy} - F_xF_yF_{yy})(F_y^2F_{xx} - 2F_xF_yF_{xy} + F_x^2F_{yy}) \\ = {} & F_y^4F_{xx}F_{xy} - 2F_y^3F_xF_{xy}^2 - F_y^3F_xF_{xx}F_{yy} + F_y^3F_xF_{xx}^2 - 3F_y^2F_x^2F_{xx}F_{xy} \\ & + 3F_y^2F_x^2F_{xy}F_{yy} + F_yF_x^3F_{xx}F_{yy} + 2F_yF_x^3F_{xy}^2 - F_yF_x^3F_{yy}^2 - F_x^4F_{xy}F_{yy}. \end{aligned}$$

From this results

THEOREM 1. At a point of either maximum or minimum curvature the relation $C = 3D$ must hold.

Having found the condition which holds at maximum and minimum points, the nature of the corresponding point on the evolute will be investigated. The equation of the evolute will be written in parametric form, $x$ being regarded as the parameter:

$$\alpha = x - \frac{F_xF_y^2 + F_x^3}{B}, \qquad \beta = y - \frac{F_x^2F_y + F_y^3}{B}.$$

The curvature, $K_1$, of the evolute is given by the formula

$$K_1 = \frac{d^2\beta}{d\alpha^2}\Big/\left[1 + \left(\frac{d\beta}{d\alpha}\right)^2\right]^{3/2}$$

and is readily computed when it is remembered that

$$d\beta/d\alpha = F_y/F_x$$

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[1] Cf. Goursat-Hedrick, *A Course in Mathematical Analysis*, vol. I, p. 438.

and therefore

$$\frac{d^2\beta}{d\alpha^2}=\frac{d\left(\dfrac{d\beta}{d\alpha}\right)}{dx}\cdot\frac{1}{\dfrac{d\alpha}{dx}}.$$

After some computation, it is found that

$$\frac{d\alpha}{dx}=\frac{F_x(C-3D)}{F_y\cdot B^2}. \tag{1}$$

Likewise

$$\frac{d\beta}{dx}=\frac{C-3D}{B^2} \tag{2}$$

and

$$\frac{d^2\beta}{d\alpha^2}=\frac{B^3}{F_x^3}\cdot\frac{1}{C-3D}.$$

Therefore

$$K_1=\frac{B^3}{(F_x^2+F_y^2)^{3/2}}\cdot\frac{1}{C-3D}. \tag{3}$$

From (1), (2), (3) it is seen that when $d\alpha/dx$ and $d\beta/dx$ vanish, $K_1$ becomes infinite and a cusp occurs on the evolute.

THEOREM 2. When the curvature of $T$ is either a maximum or a minimum (at a point where neither $F_x$ nor $F_y$ is infinite) the relation $C = 3D$ must be satisfied. Moreover, the evolute of the normals has a cusp at the same time and its curvature is infinite. Furthermore, since at a point of maximum curvature the radius of curvature is a minimum, it is seen that the corresponding point on the evolute is nearer $T$ than at any other time and that the cusp points *towards* $T$. In like manner, at a point of minimum curvature the cusp points *away* from $T$.

As a simple example consider the ellipse.

Here

$$F(x, y) \equiv b^2x^2 + a^2y^2 - a^2b^2 = 0$$

and

$$\begin{array}{lll} F_x = 2b^2x, & F_y = 2a^2y, & F_{xy} = 0, \\ F_{xx} = 2b^2, & F_{yy} = 2a^2, & F_{xxy} = 0, \\ F_{xxx} = 0, & F_{yyy} = 0, & F_{xyy} = 0, \end{array}$$

from which

$$C = 0, \qquad D = -\,64a^6b^6c^2xy$$

and

$$C - 3D = 192a^6b^6c^2xy.$$

This vanishes only when $x = 0$ or $y = 0$, and verifies the fact that maximum and minimum curvature occurs only at the vertices of the ellipse.

From the well-known form of the evolute of the ellipse,

$$(a\alpha)^{2/3} + (b\beta)^{2/3} = (a^2 - b^2)^{2/3},$$

the curvature, $K_1$, is found to be

$$K_1 = -\frac{a^{2/3}b^{2/3}(\alpha - \beta)}{3\beta^{2/3}\alpha^{2/3}(b^{4/3}\alpha^{2/3} + a^{4/3}\beta^{2/3})^{3/2}},$$

and this becomes infinite only when $\alpha = 0$ or $\beta = 0$. Thus cusps occur on the evolute at the points $\left(\pm\frac{c^2}{a}, 0\right)$, $\left(0, \pm\frac{c^2}{b}\right)$ corresponding to points of maximum and minimum curvature on the ellipse.

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# RECENT PUBLICATIONS.

## REVIEWS.

*Exercises in Algebra (including trigonometry).* By T. P. NUNN. 2 volumes. London and New York, Longmans, 1913–1914. Vol. I, 11 + 421 pp. Vol. II, 11 + 551 pp. Price 4 + 6½ shillings.

*The Teaching of Algebra (including trigonometry).* By T. P. NUNN. London and New York, Longmans, 1914. 12mo, 14 + 616 pp. Price 7½ shillings.

Among the contributions that have recently been made to the solution of the problem of teaching secondary and collegiate mathematics, Dr. Nunn's work is one of the most significant. His plan is to build up a course which shall combine the most vital and essential parts of algebra, trigonometry, analytic geometry, and elementary calculus, and present them in a way which shall meet the needs both of the student who is studying mathematics merely for its cultural, disciplinary, or informational value, and of the student who plans to go on to further study of the subject. Many teachers have serious doubts as to the wisdom of giving the same work to these two different classes of students; but Dr. Nunn's course seems to the reviewer to furnish a successful demonstration that the thing can be done.

The two volumes of *Exercises* are designed for the student, and the third volume, on *Teaching,* indicates to the teacher the reasons for the order in which topics are taken up, and gives suggestions as to methods of presentation. The first volume of the *Exercises* contains little detailed explanation for the student, this being left for the teacher to give verbally on lines indicated clearly in the *Teaching.* The second volume of the *Exercises,* however, contains more discussion of the topics treated, with the object of developing in the student the ability to read and understand mathematical writing.

The work in elementary algebra is centered about two topics, the graph and the formula. By making skillful and persistent use of both, the author gives a vitality and meaning to the fundamental laws of operation which the student can rarely gain through the mere study of the axioms and formal laws. To give one illustration, the distributive law of multiplication is not first *stated,* and then